ME 3150 Introduction to Dynamics
Exam II
Time allotted 50 minutes
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Name:____________________________________

Do all work on these papers. Question 1 50%, Question 2 50%.
1. A projectile launched from the ground with speed \( v_0 \) at an angle to the horizontal \( \theta_0 \) moves on a parabolic path relative to an earth fixed observer, \( E_O \), given by

\[
y = \tan \theta_0 x - \frac{g_e}{2(v_0 \cos \theta_0)^2} x^2
\]

Using this formula show that the maximum altitude occurs at the range \( x^* = \frac{v_0^2 \sin \theta_0 \cos \theta_0}{g_e} \) and is therefore

\[
y^* = \frac{(v_0 \sin \theta_0)^2}{2g_e}
\]

Show also that the distance travelled \( R = 2x^* \). What initial angle (in degrees) is required to produce a trajectory for which the maximum altitude is equal to the distance travelled?

**Solution.**

The Maximum altitude is the place where \( \frac{dy}{dx} = 0 \). On differentiating the expression for the trajectory as given, we get

\[
\frac{dy}{dx} = \tan \theta_0 - \frac{g_e}{(v_0 \cos \theta_0)^2} x
\]

Setting this to zero at \( x = x^* \) gives

\[
0 = \tan \theta_0 - \frac{g_e}{(v_0 \cos \theta_0)^2} x^*
\]

and solving this for \( x^* \)

\[
x^* = \frac{v_0^2 \cos \theta_0^2 \sin \theta_0}{g_e \cos \theta_0} = \frac{v_0^2 \sin \theta_0 \cos \theta_0}{g_e}
\]

as required. Substituting this value into the trajectory equation gives \( y^* \) as

\[
y^* = \frac{v_0^2 \sin \theta_0 \cos \theta_0 \sin \theta_0}{g_e \cos \theta_0} - \frac{g_e}{2(v_0 \cos \theta_0)^2} \frac{v_0^4 \sin^2 \theta_0 \cos^2 \theta_0}{g_e^2} = \frac{(v_0 \sin \theta_0)^2}{2g_e}
\]

as required. The range when \( x = R \) corresponds to \( y = 0 \). This is from the trajectory equation

\[
0 = \tan \theta_0 R - \frac{g_e}{2(v_0 \cos \theta_0)^2} R^2
\]

Solving this for \( R \) gives

\[
R = \frac{\sin \theta_0}{\cos \theta_0} \frac{2(v_0 \cos \theta_0)^2}{g_e} = \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g_e} = 2x^*
\]

The condition \( y^* = R \) corresponds to

\[
\frac{(v_0 \sin \theta_0)^2}{2g_e} = \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g_e}
\]

On cancelling and rearranging this is \( \tan \theta_0 = 4 \), which produces

\[
\theta_0 = \tan^{-1}(4) = 76.0 \text{ deg}
\]
2. A spacecraft is launched from the ground to rendezvous with a skylab orbiting in a circular orbit at an altitude of 1040 miles. The spacecraft has reached at burnout an altitude of 40 miles and a velocity of magnitude (relative to the geocentric observer $S_E$) 20000 ft/s. What is the angle $\phi_0$ that $\vec{v}_{B/E}$ should form with the local vertical $EB$ if the trajectory of the spacecraft is to be tangent at $A$ to the skylab orbit? The gravitational mass and radius of the earth are $1.409 \cdot 10^{16}$ ft$^3$/s$^2$ and 3960 miles respectively.

Solution.
In this problem the apogee is known, $r_+ = a(1 + \epsilon) = 5000$ mi, and the orbital parameters are given by

$$a = \frac{r_0}{2 - \frac{v_0^2 r_0}{M_E}} \quad \epsilon^2 = 1 - \frac{(v_{\theta 0} r_0)^2}{M_E a}$$

where $v_{\theta 0} = v_0 \sin \phi_0$. We can eliminate $\epsilon$ from these equations by equating

$$\left(\frac{r_+}{a} - 1\right)^2 = 1 - \frac{v_0^2 r_0}{M_E \ a} \sin^2 \phi_0$$

Expanding the left hand side and writing $v_0^2 r_0 / M_E = 2 - (r_0/a)$ we have

$$-\frac{2r_+}{a} + \left(\frac{r_+}{a}\right)^2 = -\left(2 - \frac{r_0}{a}\right) \frac{r_0}{a} \sin^2 \phi_0$$

Thus we find that

$$\sin \phi_0 = \sqrt{\frac{(r_+/a)(2 - r_+/a)}{(r_0/a)(2 - r_0/a)}}$$

Now

$$\frac{r_0}{a} = 2 - \frac{(4 \cdot 10^8 \text{ ft}^2/\text{s}^2)(4 \cdot 10^6 \text{ mi})(5.280 \text{ ft}/\text{mi})}{1.409 \cdot 10^{16} \text{ ft}^3/\text{s}^2} = 1.4004$$

and $r_+/a = (r_+/r_0)(r_0/a) = (5/4)(1.4004) = 1.7505$. Then

$$\sin \phi_0 = \sqrt{\frac{1.7505(2 - 1.7505)}{1.4004(2 - 1.4004)}} = 0.7212$$

Then finally the angle is $\phi_0 = \sin^{-1}(0.7212) = 46.2$ deg.