A Simple Expression for the Equation of Time

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Introduction

The equation of time quantifies the difference between time measured by a sundial and time measured by a mechanical clock. That this difference exists has been known since astronomers began making these measurements and comparing them. Moreover, since Newton showed how planetary orbits could be analyzed mathematically, it has been possible to calculate the difference as well. However, the calculation is complex and can not be represented by a simple, explicit analytical expression. The purpose of this work is to provide some asymptotic approximations to the basic relations and show how they lead to a simple yet accurate expression for the equation of time.

Solar Time and Sidereal Time

An observer fixed on the earth sees the sun and stars moving from east to west across the sky as a result of the rotation of the earth about its polar axis. The angular position of these celestial bodies is measured with respect to the meridian, which is that great circle passing through both the pole and the observer's local vertical. We call the angle between the great circle passing through the pole and the body, and the observer's meridian, the hour angle of the body. As the hour angle increases we say that time passes.

Now the sun, on any two successive transits of the meridian (this time interval defines a solar day), does not appear in the same position relative to the vernal equinox (a point in the fixed starfield). On its second transit the sun will have advanced eastward along the plane of the ecliptic (see Fig. 1.) by almost one degree, so that after approximately $n = 365.24$ such transits (this time interval defines a tropical year) it will have returned to its original position. The angle between the sun and the meridian, the hour angle of the sun H, is proportional to the local solar time and satisfies the equation

$$H = \mathrm{LST} - \alpha \tag{1}$$

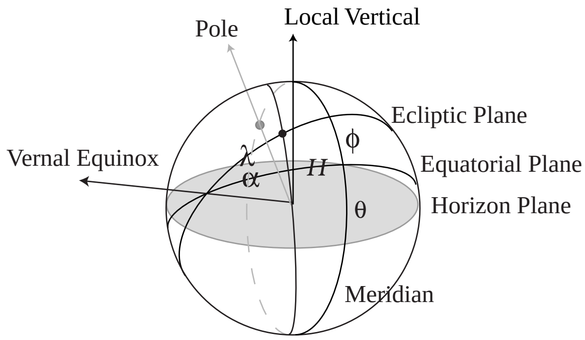

Figure 1: Geometry on the celestial sphere as seen by an observer fixed on the earth at latitude $90° - \theta$. The sun is the point on the ecliptic at latitude (the angular distance from the vernal equinox in the ecliptic plane) λ.

where the angle LST, proportional to the local sidereal time, is the hour angle of the vernal equinox, while α is the right ascension of the sun. Note that, as shown in Fig. 1., α, is the angle between the great circle through the pole and the vernal equinox, and the great circle through the pole and the sun. The hour angle, LST, which is not shown in Fig. 1., is the angle between the great circle through the pole and vernal equinox, and the meridian (two successive transits of the meridian by the vernal equinox defines a sidereal day). Equation (1) follows from Fig. 1. on noting that both hour angles increase toward the west while the right ascension increases toward the east.

We introduce solar time t_S and sidereal time, t, by making them proportional to H and LST respectively. In equation form $t_S = k_{tS}H$ and $t = k_t$ LST, where k_{tS} and k_t are dimensional constants whose dimensions are time divided by angle $[t/\angle]$. These constants are evaluated through the definition of time standards; for example, $t_S = t_{SD} = 1$ solar day when $H = 2\pi k_\angle$, and $t = t_D = 1$ sidereal day when LST $= 2\pi k_\angle$. Substituting these values into the definitions produces

$$t_S = \frac{H}{2\pi k_\angle} t_{SD} \qquad t = \frac{\text{LST}}{2\pi k_\angle} t_D \tag{2}$$

Note that $k_\angle = 1$ rad $= 180/\pi$ deg $= 1/2\pi$ rev is the dimensional constant that defines angle in terms of the ratio of the arc length of a circle to its radius. Then in terms of the times and standards defined in Eqs (2), Eq (1) can be written as

$$2\pi k_\angle \frac{t_S}{t_{SD}} = 2\pi k_\angle \frac{t}{t_D} - \alpha \tag{3}$$

This equation relates solar time to sidereal time. For $t = t_D$, we get from substitution, $t_S = (1 - \alpha/2\pi k_\angle)t_{SD}$. Since α varies nonuniformly throughout the year, this says that each sidereal day corresponds to a different number of solar days.

In one tropical year $t_S \equiv t_Y = n\, t_{SD}$, the sun completes one circuit on the ecliptic, $\alpha = 2\pi k_\angle$, so we get from substituting into Eq (3) and solving, $t \equiv t_Y = (n+1)t_D$. In this way we get a conversion equation between the scales of the two time measures

$$t_Y = 1\text{yr} = nt_{SD} = (n+1)t_D \tag{4}$$

We see from Eq (4) that solar time runs slower than sidereal time since a solar second is longer than a sidereal second. Indeed, Eq (4) tells us that in one year sidereal time accumulates an entire day, 24 hours, more than solar time. Based on this, we define an average, 1 mean solar day = t_{MSD} = (1/365.24) year = t_Y/n.

We can rewrite Eq (3) by using Eq (4) to eliminate both t_D and t_{SD}. This gives

$$2\pi n k_\angle \frac{t_S}{t_Y} = 2\pi(n+1)k_\angle \frac{t}{t_Y} - \alpha = 2\pi k_\angle \left(n\frac{t}{t_Y} + \frac{t}{t_Y} \right) - \alpha$$

Then using the earth's rotation rate in the form, $f_E = 2\pi n k_\angle / t_Y = 2\pi k_\angle / t_{MSD}$, we get

$$t_S - t = \Delta t = \frac{2\pi k_\angle t/t_Y - \alpha}{f_E} \tag{5}$$

Equation (5) is the equation of time; in it both t_S and t are measured in the same units, mean solar days. Accordingly, we note that if α increased linearly with time, $\alpha = 2\pi k_\angle t/t_Y$, there would be no more difference between solar time and sidereal time than there is between a foot and a meter. However, as we mentioned before this is not the case; α varies throughout the year due to the sun's nonuniform motion around the earth (as seen by an earth fixed observer) in the plane of the ecliptic and due to the difference between the equatorial and ecliptic planes. These effects have been quantified by measurements of $\alpha(t)$ by many astronomers from ancient times to the present. Their results are readily available in the form of tables and graphs. In addition to evaluating Δt by measurements, it can also be calculated by evaluating $\alpha(t)$ from the equations for planetary motion that follow from Newton's laws.

Note that the first term on the right side of Eq (5) is the time indicated by a theoretical mean sun; this is the time measured by a mechanical clock that beats 24 hours in one mean solar day. The second term is the corresponding time indicated by the true sun, which is the time measured by a sundial.

Calculation of the Sun's Right Ascention

All the angles we have written till now were physical angles that are specified by the product of a number and an angular unit. This is the case with all dimensional quantities, but angle is a derived dimension that is normally used in dimensionless form. For example, we write $\alpha = \tilde{\alpha}$ rad where α is the physical angle and $\tilde{\alpha}$ is its dimensionless equivalent. Now the arguments of all trigonometric functions must be dimensionless angles; however, in order to keep from proliferating symbols, we will not write the tilde over the angle symbol, but understand that all angles that appear in our equations from now on will be dimensionless. Thus we rewrite Eq (5) by dividing both its numerator and denominator by $k_\angle$

$$\Delta t = \frac{2\pi t/t_Y - \alpha}{\omega_E} \tag{6}$$

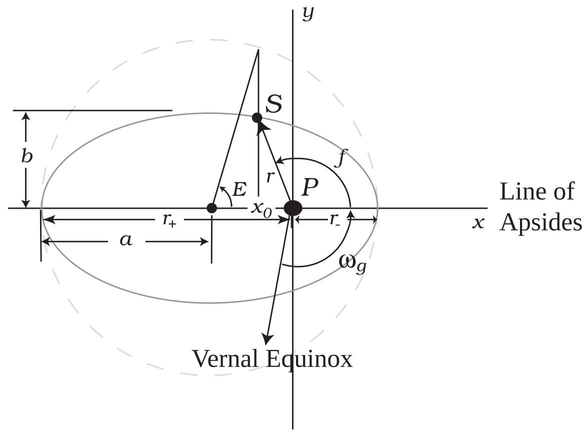

Figure 2: The geometry of the sun's elliptical orbit relative to a non rotating earth fixed observer P in the ecliptic plane.

where $\omega_E = 2\pi n/t_Y = 2\pi/t_{MSD}$ is the angular speed of the earth and α is now the dimensionless angle corresponding to its physical counterpart divided by $k_\angle$. Whenever we express a physical angle in radians, the dimensionless angle is just the numerical factor.

If we consider the right spherical triangle shown in Fig. 1. with sides α and λ and included angle ϕ, we find on using the appropriate relations that apply to spherical triangles

$$\tan\alpha = \cos\phi\tan\lambda \tag{7}$$

The equation that specifies clock time as a function of the angular location of the sun in its orbit is

$$M = E - \epsilon\sin E \tag{8}$$

and is called Kepler's equation. The angle E, called the eccentric anomaly, is shown in Fig. 2., the angle $M = 2\pi(t - t_-)/t_Y$, called the mean anomaly, measures the dimensionless angular distance from perigee that would be travelled by a uniformly moving (mean) sun in the given time interval, and $\epsilon = (b-a)/a$ is the eccentricity of the elliptical path. The eccentric anomaly is not measurable by an observer on the earth, but it is related to the measurable angle f between the line of apsides and the sun (see Fig. 2.). Of the many relations between these angles the one that is useful to us is

$$\tan\frac{E}{2} = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{f}{2} \tag{9}$$

Equations (8) and (9) both result from Newton's analysis of planetary motion.

The final equation that is needed relates λ and f. It is

$$\lambda = f + \omega_g \tag{10}$$

and can be verified by comparing Figs. 1. and 2. The four equations, Eqs (7) - (10), specify the four variables, E, f, λ, and α, as functions of M, along with the parameters $\phi = 0.40910$, $\epsilon = 0.0167$, and $\omega_g = 4.9358$.

If we note that $\omega_g = 2\pi t_-/\omega_E$, where t_- is the time the perigee is passed counting zero from the vernal equinox, we can write the equation of time, Eq (6), in the form

$$\Delta t = \frac{M + \omega_g - \alpha}{\omega_E} \tag{11}$$

Using the fact that both ϵ and ϕ are small, we can make some approximations that produce a simple expression for Δt while still retaining enough accuracy to be useful.

Asymptotic Analysis of the Equation of Time

If we write Eq (8) in the form $E = M + \epsilon \sin E$ and account for the fact that $\epsilon << 1$, an approximate solution is $E \sim M$. This can be improved by iteration, namely by substituting this approximation into the right hand side. This results in

$$E \sim M + \epsilon \sin M \tag{12}$$

This could be improved further by iterating again, but will not be necessary for our purposes.

Equation (9) can also be simplified because ϵ is small. Solving for f and expanding the ϵ function in a power series produces (keeping two terms in ϵ)

$$f \sim 2 \tan^{-1}[\tan(E/2) + \epsilon \tan(E/2)] \tag{13}$$

This function can itself be expanded in a power series in ϵ. We get for the first two terms of this, on recalling that $d \tan^{-1} x/dx = 1/(1 + x^2)$,

$$f \sim E + \epsilon \sin E \tag{14}$$

Substituting Eq (12) into Eq (14) and keeping only terms linear in ϵ, we get

$$f \sim M + 2\epsilon \sin M$$

This approximation has been known since Kepler's time; it is called the equation of the center. Using this result in Eq (10) we find

$$\lambda \sim M + \omega_g + 2\epsilon \sin M \tag{15}$$

Now on noting that the two term expansion $1 - \phi^2/2$ of $\cos \phi$ differs from it by about 0.1 percent for its actual value $\cos \phi = 0.40910$, we write Eq (7) in the same form as Eq (13)

$$\alpha \sim \tan^{-1}[\tan \lambda - (\phi^2/2) \tan \lambda]$$

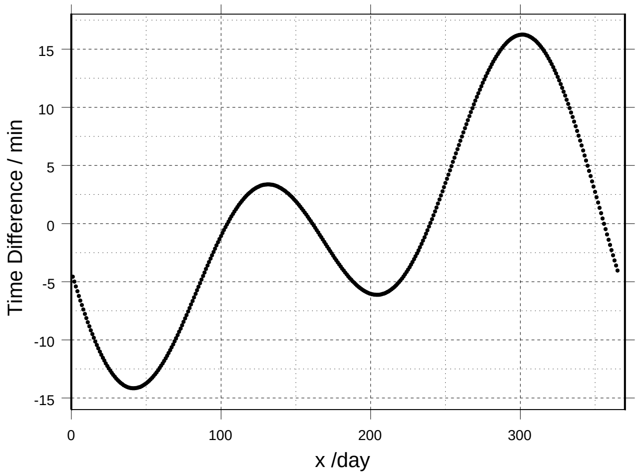

Figure 3: A plot of Eq (18). The abscissa, $x/\text{day} = t - t_-/\text{day}$, is the number of clock time mean solar days counted from perigee.

Then the expansion of this is, just as we got in Eq (14)

$$\alpha \sim \lambda - \frac{\phi^2}{4}\sin(2\lambda) \tag{16}$$

Substituting Eq (15) into Eq (16) and retaining only terms linear in ϵ and ϕ^2, we have when we put this result into Eq (11)

$$\Delta t \sim \frac{-2\epsilon\sin M + 0.25\,\phi^2\sin(2M + 2\omega_g)}{\omega_E} \tag{17}$$

This is an approximate, but accurate and easily calculable, expression for the equation of time. Substituting numerical values into Eq (17) produces the numerical equation

$$\frac{\Delta t}{\text{min}} \sim 229.18\left[-0.0334\sin\left(\frac{2\pi}{365.24}\frac{t-t_-}{\text{day}}\right) + 0.04184\sin\left(\frac{4\pi}{365.24}\frac{t-t_-}{\text{day}} + 3.5884\right)\right] \tag{18}$$

A plot of this function is shown in Fig. 3. It agrees rather well with more exact numerical solutions.

References

[1] Duffett-Smith P 1988 *Practical Astronomy with your Calculator, Third Edition* (Cambridge: Cambridge University Press)

[2] Moulton F R 1970 *An Introduction to Celestial Mechanics, Second Revised Edition* (New York: Dover)